Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER
Journal of Approximation Theory 127 (2004) 135–154

Journal of Approximation Theory
http://www.elsevier.com/locate/jat

Radial basis functions under tension

A. Bouhamidi[a,*] and A. Le Méhauté[b]

[a] LMPA, Université du Littoral Côte d'Opale, 50 rue F. Buisson BP 699, F-62228 Calais Cedex, France
[b] Laboratoire de Mathématiques, CNRS UMR 6629, Université de Nantes, Faculté des Sciences, 2, rue de la Houssinière, F-44072 Nantes Cedex, France

Received 27 February 2003; accepted in revised form 3 March 2004

Communicated by Martin Buhmann

Dedicated to the memory of Will Light

Abstract

We discuss multivariate interpolation with some radial basis function, called radial basis function under tension (RBFT). The RBFT depends on a positive parameter which provides a convenient way of controlling the behavior of the interpolating surface. We show that our RBFT is conditionally positive definite of order at least one and give a construction of the native space, namely a semi-Hilbert space with a semi-norm, minimized by such an interpolant. Error estimates are given in terms of this semi-norm and numerical examples illustrate the behavior of interpolating surfaces.

MSC: 65D05; 65D07; 65D15; 41A15

Keywords: Radial basis function; Conditionally positive definite functions; Spline under tension; Multivariate interpolation

1. Introduction

Multivariate interpolation and approximation with radial basis functions have been comprehensively reviewed in several recent papers (see [4,16,17] among others), and it is sufficient here to roughly mention how it works.

*Corresponding author.
E-mail addresses: a.bouhamidi@lmpa.univ-littoral.fr (A. Bouhamidi), alm@math.univ-nantes.fr (A. Le Méhauté).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.03.005

A radial function Φ on $\mathbb{R}^d$ is defined through a univariate function $\phi : [0, \infty) \rightarrow \mathbb{R}$ in such a way that $\Phi(x) = \phi(||x||^2)$ where $||\,.\,||$ is the usual Euclidean norm in $\mathbb{R}^d$. Interpolation at N scattered points in $\mathbb{R}^d$ can be carried out using translates of the function Φ to generate a linear space of interpolating functions. The usual radial basis functions are thin plate splines [5] where $\phi(t) = t \ln \sqrt{t}$, Hardy's multiquadrics [10] with $\phi(t) = \sqrt{t + c^2}$ and inverse multiquadrics with $\phi(t) = (\sqrt{t + c^2})^{-1}$, as well as the Gaussians with $\phi(t) = e^{-c^2 t}$.

This paper discusses interpolation of real-valued functions on a finite set of scattered data in $\mathbb{R}^d$ by a linear combination of translates of a radial function which depends on a positive parameter that allows us to control the behavior of the interpolating surface, and gives a generalization of the well-known univariate spline under tension [19]. The problem of interpolation by a surface involving the concept of tension goes back to Franke [7,15]. His construction of the thin plate spline with tension is similar to the construction of the thin plate splines by Harder and Desmarais [9]. Unfortunately, this approach does not lead to a simple representation. On the other hand, the radial basis function we present here has an elementary representation and allows also the control over the behavior of the interpolating surface and provides nice geometrical tension effects. Our construction is based on a variational formulation, but, unlike Franke's one, we cannot assign a physical meaning for the tension parameter. For another approach of surface splines with tension which models a physical process, but with a more complicated basis function, we refer to [1–3].

In Section 2 we briefly review some properties of spline curves under tension, with a new formulation, as they are considered here as univariate radial basis functions, providing a slight modification of the usual ones [19]. In Section 3 we discuss some generalizations to the multi-dimensional case and the variational problem with the construction of the corresponding semi-Hilbert space with minimization of the semi-norm. The limit cases and error estimates are given in Section 4. We conclude this paper by some numerical examples illustrating the effect of tension and the performances of the interpolant for surfaces.

2. Splines curves under tension

We first briefly review some properties concerning univariate splines under tension (for more details on this approach, see [1–3]). This presentation differs quite a bit from the usual definition, but provides a framework for a generalization to higher dimension, as we will see thereafter.

Let $\mathscr{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$, let $\tau > 0$ be a real number and let $\mathscr{H}$ be the subspace of $\mathscr{D}'(\mathbb{R})$

$$\mathscr{H} = \{u \in \mathscr{D}'(\mathbb{R});\ u' \text{ and } u'' \in L^2(\mathbb{R})\} \tag{2.1}$$

endowed with the semi-scalar product

$$(u|v)_{\mathscr{H}} = \int_{-\infty}^{+\infty} u''(t)v''(t)\,dt + \tau^2 \int_{-\infty}^{+\infty} u'(t)v'(t)\,dt, \tag{2.2}$$

and the associated semi-norm $|u|_{\mathscr{H}}$. The real parameter τ is called a *tension parameter*. Let $\mathscr{C}(\mathbb{R})$ and $\Pi_0(\mathbb{R})$ be the space of continuous functions and the space of constant polynomials on $\mathbb{R}$, respectively. The space $\mathscr{H}$ is a semi-Hilbert subspace of $\mathscr{C}(\mathbb{R})$ (see [1–3]) and its null-space is $\Pi_0(\mathbb{R})$.

Let $\mathscr{D}_\tau$ be the linear differential operator defined on $\mathscr{D}'(\mathbb{R})$ by $\mathscr{D}_\tau(.) = \frac{d^4(.)}{dt^4} - \tau^2\frac{d^2(.)}{dt^2}$. For any $u \in \mathscr{H}$ and any compactly supported infinitely differentiable function φ on $\mathbb{R}$, we have $(u|\varphi)_{\mathscr{H}} = \langle \mathscr{D}_\tau u, \varphi \rangle = \langle u, \mathscr{D}_\tau \varphi \rangle$. A fundamental solution Φ of the linear differential operator $\mathscr{D}_\tau$ is a tempered distribution on $\mathbb{R}$ such that, in the distributional sense

$$\mathscr{D}_\tau \Phi = \delta, \tag{2.3}$$

as usual, δ denotes the Dirac's measure at the origin. It is obvious that such a fundamental solution cannot be unique (at least up to a polynomial of degree 1). It is easy to find out that a fundamental solution for the linear differential operator $\mathscr{D}_\tau$ is

$$\Phi(t) = -\frac{1}{2\tau^3}(e^{-\tau|t|} + \tau|t|). \tag{2.4}$$

Let us notice that Φ is a function of class $\mathscr{C}^2(\mathbb{R})$ which generates a tempered distribution that we still denote by Φ. We have [1–3]:

Theorem 1. *Let $\{(t_1,f_1), \ldots, (t_N,f_N)\} \subset \mathbb{R}^2$ be any set of real data with $t_i \neq t_j$ for $i \neq j$, let Φ be the function given by* (2.4). *Then, there is a unique function $s_\tau \in \mathscr{H}$ minimizing the semi-norm $|\,.\,|_{\mathscr{H}}$ of $\mathscr{H}$, subject to the interpolation constraints $s_\tau(t_i) = f_i$ for $i = 1, \ldots, N$. The function s_τ has explicitly the following form*:

$$s_\tau(t) = \sum_{i=1}^{N} \lambda_i \Phi(t - t_i) + \lambda_{N+1}, \tag{2.5}$$

where the coefficients $\lambda_1, \ldots, \lambda_{N+1}$ are solutions of the nonsingular linear system

$$\begin{pmatrix} \Phi(t_1 - t_1) & \cdots & \Phi(t_1 - t_N) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \Phi(t_N - t_1) & \cdots & \Phi(t_N - t_N) & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \\ \lambda_{N+1} \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \\ 0 \end{pmatrix}. \tag{2.6}$$

Let $\mathscr{C}'(\mathbb{R})$ be the dual space of $\mathscr{C}(\mathbb{R})$, namely the space of compactly supported Radon measures and let $\Pi_0^\perp(\mathbb{R})$ be the subspace orthogonal to $\Pi_0(\mathbb{R})$ in the distributional sense. The semi-Hilbertian kernel of $\mathscr{H}$ is the mapping $H : \mathscr{C}'(\mathbb{R}) \cap \Pi_0^\perp(\mathbb{R}) \to \mathscr{H}$ such that $H(\mu) = \mu * \Phi$ for $\mu \in \mathscr{C}'(\mathbb{R}) \cap \Pi_0^\perp(\mathbb{R})$. According to

[18], this semi-Hilbertian kernel satisfies

$$\langle \mu, \mu * \Phi \rangle = \int \int \Phi(t-s)\, d\mu(t)\, d\mu(s) \geqslant 0,$$

for all Radon measures μ compactly supported and such that $\int d\mu(t) = 0$.

Specifically, given any set of p distinct points $x_1, \dots, x_p$ in $\mathbb{R}$ and any discrete measure $\mu = \sum_{i=1}^{p} c_i \delta_{x_i}$ such that $\sum_{i=1}^{p} c_i = 0$ (δ_{x_i} is the Dirac's measure at x_i), we have $\sum_{i=1}^{p} \sum_{j=1}^{p} c_i c_j \Phi(x_i - x_j) \geqslant 0$. Therefore (see Section 3), the function Φ is a conditionally positive definite function on $\mathbb{R}$ of order 1.

3. Multi-dimensional case

Let $m \geqslant 1$ be a positive integer and let $\Pi_{m-1}(\mathbb{R}^d)$ denote the space of polynomials on $\mathbb{R}^d$ of degree at most $m-1$ whose dimension is denoted $d(m)$.

Let us give an arbitrary finite set $\mathscr{A} = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$ of distinct interpolation points and a set of real data $\{f_1, \dots, f_N\}$. We assume that $N \geqslant d(m)$ and that $\mathscr{A}$ contains a $\Pi_{m-1}(\mathbb{R}^d)$–unisolvent set (i.e., if $p \in \Pi_{m-1}(\mathbb{R}^d)$ and if for all $a \in \mathscr{A}$, $p(a) = 0$, then $p \equiv 0$). Let $(q_1, \dots, q_{d(m)})$ be a basis of $\Pi_{m-1}(\mathbb{R}^d)$ and $||\,.\,||$ be the Euclidean norm on $\mathbb{R}^d$. We consider the following function:

$$\phi_\tau(t) = -\frac{1}{2\tau^3}(e^{-\tau\sqrt{t}} + \tau\sqrt{t}), \quad t \in \mathbb{R}^+, \tag{3.1}$$

and the following radial function

$$\Phi_\tau(x) = \phi_\tau(||x||^2) = -\frac{1}{2\tau^3}(e^{-\tau||x||} + \tau||x||), \quad x \in \mathbb{R}^d. \tag{3.2}$$

Let

$A_\tau = (\Phi_\tau(x_i - x_j))_{1 \leqslant i,j \leqslant N}$ be a $N \times N$ matrix,
$M = (q_j(x_i))_{\substack{1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant d(m)}}$ be a $N \times d(m)$ matrix, M^T denotes the transpose of M,
O be the $d(m) \times d(m)$ zero matrix,
$\lambda = (\lambda_1, \dots, \lambda_N)^T$ and $f = (f_1, \dots, f_N)^T$ two vectors of $\mathbb{R}^N$,
and $\alpha = (\alpha_1, \dots, \alpha_{d(m)})^T$ be a vector of $\mathbb{R}^{d(m)}$.

We consider a function $s_{\mathscr{A},\tau,m}$ of the following form:

$$s_{\mathscr{A},\tau,m}(x) = \sum_{i=1}^{N} \lambda_i \Phi_\tau(x - x_i) + \sum_{j=1}^{d(m)} \alpha_j q_j(x), \quad x \in \mathbb{R}^d. \tag{3.3}$$

The interpolation conditions $s_{\mathscr{A},\tau,m}(x_j) = f_j$, for $j = 1, \dots, N$ together with the constraint $\sum_{i=1}^{N} \lambda_i p(x_i) = 0$, for all $p \in \Pi_{m-1}(\mathbb{R}^d)$ provide a linear system, similar to

(2.6) obtained for the univariate case. The coefficients λ_i and α_i are solutions of

$$\begin{pmatrix} A_\tau & M \\ M^T & O \end{pmatrix}\begin{pmatrix} \lambda \\ \alpha \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \text{i.e.} \quad \begin{cases} A_\tau\lambda + M\alpha &= f, \\ M^T\lambda &= 0. \end{cases} \tag{3.4}$$

Remark 1. The factor $\frac{1}{2\tau^3}$ which appear in (3.1) and (3.2) may be omitted, because it will be absorbed in the coefficients of the first linear combination of the right-hand side of (3.3). In one dimension this factor is only obtained to satisfy condition (2.3). Furthermore, according to our numerical experience, the RBFTs have the same behavior using the radial function Φ_τ with or without the factor $\frac{1}{2\tau^3}$. In this paper, we consider the radial function Φ_τ as it is written in (3.2), i.e. with the factor $\frac{1}{2\tau^3}$.

From [14], a sufficient condition for the linear system (3.4) to be nonsingular for any set of interpolation data points is for Φ_τ to be a conditionally strictly positive definite function of order m on $\mathbb{R}^d$. We recall the following

Definition 1. A continuous real valued function F defined on $\mathbb{R}^d$ is conditionally positive definite of order m on $\mathbb{R}^d$, if for any positive integer p, any set of points $x_1, \dots, x_p \in \mathbb{R}^d$ and any real-vector $c = (c_1, \dots, c_p) \in \mathbb{R}^p$ such that $\sum_{i=1}^p c_i x_i^\alpha = 0$ for all $|\alpha| < m$, we have

$$\sum_{i=1}^{p}\sum_{j=1}^{p} c_i c_j F(x_i - x_j) \geqslant 0. \tag{3.5}$$

When the right-hand side of (3.5) is >0, with $c \neq 0$ and for any configuration of points, the function F is conditionally strictly positive definite of order m on $\mathbb{R}^d$.

The following theorem of Micchelli [14] (completed by Guo et al. [8]) gives a nice characterization for radial functions to be conditionally positive definite on $\mathbb{R}^d$ for all integer $d \geqslant 1$:

Theorem 2 (Guo et al. [8], Micchelli [14]). *Given $g : [0, \infty) \to \mathbb{R}$, the radial function $G(x) = g(||x||^2)$ is a conditionally strictly positive definite function of order m on $\mathbb{R}^d$ for all integer $d \geqslant 1$ if and only if $g \in \mathscr{C}[0, \infty) \cap \mathscr{C}^\infty(0, \infty)$ and*

$$(-1)^{m+k} g^{(m+k)}(t) > 0, \quad \textit{for all} \quad t > 0 \quad \textit{and} \quad k = 0, 1, 2, \dots$$

We obtain the following:

Theorem 3. *The function Φ_τ defined on $\mathbb{R}^d$ by*

$$\Phi_\tau(x) = -\frac{1}{2\tau^3}(e^{-\tau||x||} + \tau||x||)$$

is conditionally strictly positive of order $m \geqslant 1$ on $\mathbb{R}^d$ for all integer $d \geqslant 1$.

Proof. It is obvious to see that the function ϕ_τ given by (3.1) belongs to $\mathscr{C}[0,\infty)\cap\mathscr{C}^\infty(0,\infty)$. In order to obtain the result, we only have to prove that the function ϕ_τ satisfies the condition $(-1)^k\phi_\tau^{(k)}(t)>0$ for all $t>0$ and for all integer $k\geqslant 1$. Let $c_{0,1}=1$, $c_{0,2}=c_{1,2}=1$, $c_{0,3}=c_{1,3}=1$ and $c_{2,3}=\frac{2}{3}$, then for $t>0$ we successively have

$$\phi_\tau{}'(t) = -\frac{1}{2^2\tau^2 t^{1/2}e^{\tau\sqrt{t}}}[e^{\tau\sqrt{t}} - c_{0,1}],$$

$$\phi_\tau^{(2)}(t) = \frac{1}{2^3\tau^2 t^{3/2}e^{\tau\sqrt{t}}}[e^{\tau\sqrt{t}} - c_{0,2} - c_{1,2}(\tau\sqrt{t})],$$

$$\phi_\tau^{(3)}(t) = -\frac{1.3}{2^4\tau^2 t^{5/2}e^{\tau\sqrt{t}}}\left[e^{\tau\sqrt{t}} - c_{0,3} - c_{1,3}(\tau\sqrt{t}) - c_{2,3}\frac{(\tau\sqrt{t})^2}{2!}\right].$$

By induction, we suppose that for all integer $k>1$, there exists coefficients $c_{j,k}$ satisfying $0<c_{j,k}\leqslant 1$ for $j=0,\dots,k-1$, such that

$$\phi_\tau^{(k)}(t) = \frac{(-1)^k(1.3\dots(2k-3))}{2^{k+1}\tau^2 t^{(2k-1)/2}e^{\tau\sqrt{t}}}\left[e^{\tau\sqrt{t}} - \sum_{j=0}^{k-1} c_{j,k}\frac{(\tau\sqrt{t})^j}{j!}\right], \quad \forall t>0. \tag{3.6}$$

Calculating the derivative of $\phi_\tau^{(k)}$, we obtain

$$\phi_\tau^{(k+1)}(t) = \frac{(-1)^{k+1}(1.3\dots(2k-3)(2k-1))}{2^{k+2}\tau^2 t^{(2k+1)/2}e^{\tau\sqrt{t}}}\left[e^{\tau\sqrt{t}} - \sum_{j=0}^{k-1} c_{j,k}\frac{(\tau\sqrt{t})^j}{j!}\right.$$
$$\left. - \sum_{j=0}^{k-1}\frac{c_{j,k}}{2k-1}\frac{(\tau\sqrt{t})^{j+1}}{j!} + \sum_{j=1}^{k-1}\frac{c_{j,k}}{2k-1}\frac{(\tau\sqrt{t})^j}{(j-1)!}\right], \quad \forall t>0. \tag{3.7}$$

Setting

$$\begin{cases} c_{0,k+1} &= c_{0,k} = 1, \\ c_{j,k+1} &= \left(1-\dfrac{j}{2k-1}\right)c_{j,k} + \dfrac{j}{2k-1}c_{j-1,k}, \quad \text{for } 1\leqslant j\leqslant k-1, \\ c_{k,k+1} &= \dfrac{k}{2k-1}c_{k-1,k} = \dfrac{k!}{1.3\dots(2k-1)}, \end{cases}$$

expression (3.7) may be arranged in the following form:

$$\phi_\tau^{(k+1)}(t) = \frac{(-1)^{k+1}(1.3\dots(2k-1))}{2^{k+2}\tau^2 t^{(2k+1)/2}e^{\tau\sqrt{t}}}\left[e^{\tau\sqrt{t}} - \sum_{j=0}^{k} c_{j,k+1}\frac{(\tau\sqrt{t})^j}{j!}\right], \quad \forall t>0. \tag{3.8}$$

Since $0<c_{j,k}\leqslant 1$ we also obtain $0<c_{j,k+1}\leqslant 1$. Thus $(-1)^k\phi_\tau^{(k)}(t)>0$, for all $t>0$, therefore Φ_τ is conditionally strictly positive definite function of order $m\geqslant 1$. $\square$

Now, we will discuss the associated variational problem with the construction of the corresponding semi-Hilbert space with minimization of the semi-norm.

Let $\mathscr{C}(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$, let $\mathscr{C}'(\mathbb{R}^d)$ denote the topological dual space of $\mathscr{C}(\mathbb{R}^d)$, which is the space of all compactly supported

Radon measures on $\mathbb{R}^d$ and let $\overset{\bullet}{E}_m = \mathscr{C}(\mathbb{R}^d)/\Pi_{m-1}(\mathbb{R}^d)$ be the quotient of the space $\mathscr{C}(\mathbb{R}^d)$ by $\Pi_{m-1}(\mathbb{R}^d)$. The quotient space $\overset{\bullet}{E}_m$ is a locally convex topological vector space and its topological dual $\overset{\bullet}{E'}_m$ is naturally isomorphic to the space of all compactly supported Radon measures which are orthogonal, in the distributional sense, to $\Pi_{m-1}(\mathbb{R}^d)$. Then both spaces can be identified and we can write

$$\overset{\bullet}{E'}_m = \left\{ \mu \in \mathscr{C}'(\mathbb{R}^d) \colon \int x^\alpha \, d\mu(x) = 0 \quad \text{for all } |\alpha| < m \right\}$$

We directly obtain from [12,13] and [18]

Theorem 4. *There is a unique semi-Hilbert subspace $\mathscr{H}_{\tau,m}$ of $\mathscr{C}(\mathbb{R}^d)$ equipped with a semi-inner product $(.\,|\,.)_{\mathscr{H}_{\tau,m}}$ and its associated semi-norm $|\,.\,|_{\mathscr{H}_{\tau,m}}$ such that*

(1) *The null-space of $(\mathscr{H}_{\tau,m}, |\,.\,|_{\mathscr{H}_{\tau,m}})$ is $\Pi_{m-1}(\mathbb{R}^d)$.*

(2) *$\overset{\bullet}{\mathscr{H}}_{\tau,m} = \mathscr{H}_{\tau,m}/\Pi_{m-1}(\mathbb{R}^d)$ equipped with the scalar product $(\overset{\bullet}{u} \,|\, \overset{\bullet}{v})_{\overset{\bullet}{\mathscr{H}}_{\tau,m}} = (u|v)_{\mathscr{H}_{\tau,m}}$ is a Hilbert space continuously embedded in $\overset{\bullet}{E}_m$.*

(3) *The mapping $H : \overset{\bullet}{E'}_m \to \overset{\bullet}{E}_m$ defined by $H\mu = \overset{\bullet}{\overbrace{\mu * \Phi_\tau}}$ is the unique Hilbertian kernel of $\overset{\bullet}{\mathscr{H}}_{\tau,m}$ with respect to $\overset{\bullet}{E}_m$. Furthermore, these properties uniquely determine $\mathscr{H}_{\tau,m}$ and $(.\,|\,.)_{\mathscr{H}_{\tau,m}}$.*

Remark 2. We note that the space $\mathscr{H}_{\tau,m}$ and its semi-inner product $(.\,|\,.)_{\mathscr{H}_{\tau,m}}$, given in Theorem 4, depend on the parameter τ, so the subscript τ attached to the notations is to emphasize this dependence. The space $\mathscr{H}_{\tau,m}$ is described in Theorem 4 using a Hilbertian kernel tool. An explicit construction of this space, as a Sobolev space type, unless for $d = 1$ is not easy to carry out here. Some investigations in this direction are in progress.

Since $\mathscr{A}$ contains a $\Pi_{m-1}(\mathbb{R}^d)-$unisolvent set, then the following bilinear form:

$$\langle u|v \rangle_{\mathscr{H}_{\tau,m}} = (u|v)_{\mathscr{H}_{\tau,m}} + \sum_{a \in \mathscr{A}} u(a)v(a), \quad \text{for} \quad u, v \in \mathscr{H}_{\tau,m} \tag{3.9}$$

is an inner product on $\mathscr{H}_{\tau,m}$. We directly obtain from Theorem 4 and the closed graph theorem, the following

Corollary 1. *The space $\mathscr{H}_{\tau,m}$ with the inner product* (3.9) *and the associated norm denoted by $||u||_{\mathscr{H}_{\tau,m}}$ is a Hilbert subspace of $\mathscr{C}(\mathbb{R}^d)$, continuously embedded in $\mathscr{C}(\mathbb{R}^d)$* (*that is $\mathscr{H}_{\tau,m}$ is a Hilbertian subspace of $\mathscr{C}(\mathbb{R}^d)$ in the sense of Schwartz*).

We can now state the following main theorem

Theorem 5. *Let N be an integer such that $N \geqslant d(m)$, let us give an arbitrary finite set $\mathscr{A} = \{x_1, \dots, x_N\}$ of distinct points in $\mathbb{R}^d$ and a set of real data $\{f_1, \dots, f_N\}$. We assume that $\mathscr{A}$ contains a $\Pi_{m-1}(\mathbb{R}^d)$–unisolvent. Then, there exists one and only one function $s_{\mathscr{A},\tau,m}$ in $\mathscr{H}_{\tau,m}$ satisfying the interpolating conditions $s_{\mathscr{A},\tau,m}(x_i) = f_i$ for $i = 1, \dots, N$ and minimizing the semi-norm $|\,.\,|_{\mathscr{H}_{\tau,m}}$. It has the form shown in* (3.3) *while the coefficients λ_i, α_j, for $i = 1, \dots, N+1$ and $j = 1, \dots, d(m)$ are solutions of the nonsingular linear system* (3.4).

Proof. (a) Existence and unicity: Since, Dirac's measure δ_{x_i} at the point x_i is a linear continuous functional over $(\mathscr{H}_{\tau,m}, ||\,.\,||_{\mathscr{H}_{\tau,m}})$, it follows that the set $I_i := \{u \in \mathscr{H}_{\tau,m} : u(x_i) = f_i\} = \delta_{x_i}^{-1}\{f_i\}$, for $i = 1, \dots, N$, is a closed subset of $(\mathscr{H}_{\tau,m}, ||\,.\,||_{\mathscr{H}_{\tau,m}})$. Then, the set $I_{\mathscr{A}} := \bigcap_{i=1}^N I_i = \{u \in \mathscr{H}_{\tau,m} : u(x_i) = f_i, i = 1, \dots, N\}$ is also a closed subset of $(\mathscr{H}_{\tau,m}, ||\,.\,||_{\mathscr{H}_{\tau,m}})$ and it is obviously a convex set. Now by the projection theorem we obtain existence and uniqueness of the projection of $0_{\mathscr{H}_{\tau,m}}$ on $I_{\mathscr{A}}$, which is the unique element of $I_{\mathscr{A}}$ of minimal semi-norm $|\,.\,|_{\mathscr{H}_{\tau,m}}$. Let us call $s_{\mathscr{A},\tau,m}$ this unique element.

(b) Form of the solution: We have $\langle s_{\mathscr{A},\tau,m} | u \rangle_{\mathscr{H}_{\tau,m}} = 0$ for all $u \in \mathscr{H}_{\tau,m}$ vanishing on $\mathscr{A}$, which also implies that $(s_{\mathscr{A},\tau,m} | u)_{\mathscr{H}_{\tau,m}} = 0$ for all $u \in \mathscr{H}_{\tau,m}$ vanishing on $\mathscr{A}$. Let us choose an arbitrary set of functions $L_i \in \mathscr{H}_{\tau,m}$, such that $L_i(x_i) = 1$ and $L_i(x_j) = 0$ for all $j \neq i$. For any $v \in \mathscr{H}_{\tau,m}$ the function $u = v - \sum_{i=1}^N v(x_i) L_i$ vanishes on $\mathscr{A}$, therefore $(s_{\mathscr{A},\tau,m} | u)_{\mathscr{H}_{\tau,m}} = 0$, and so

$$(s_{\mathscr{A},\tau,m} | v)_{\mathscr{H}_{\tau,m}} = \sum_{i=1}^N (s_{\mathscr{A},\tau,m} \,| L_i)_{\mathscr{H}_{\tau,m}} v(x_i) = \sum_{i=1}^N \lambda_i v(x_i) = \langle \mu, v \rangle, \quad \forall v \in \mathscr{H}_{\tau,m},$$

where $\mu = \sum_{i=1}^N \lambda_i \delta_{x_i}$ and $\lambda_i = (s_{\mathscr{A},\tau,m} | L_i)_{\mathscr{H}_{\tau,m}}$. The support of μ is $\mathscr{A}$ and $\langle \mu, q \rangle = (s_{\mathscr{A},\tau,m} | q)_{\mathscr{H}_{\tau,m}} = 0$, $\forall q \in \Pi_{m-1}(\mathbb{R}^d)$. Then, using Theorem 4, we can write $\langle \mu, v \rangle = (\dot{v} \,| H\mu)_{\dot{\mathscr{H}}_{\tau,m}} = (\dot{v} \,| \overbrace{\mu * \Phi_\tau}^{\bullet})_{\dot{\mathscr{H}}_{\tau,m}} = (v | \mu * \Phi_\tau)_{\mathscr{H}_{\tau,m}}$, $\forall v \in \mathscr{H}_{\tau,m}$. Therefore, $(s_{\mathscr{A},\tau,m} - \mu * \Phi_\tau | v)_{\mathscr{H}_{\tau,m}} = 0$, $\forall v \in \mathscr{H}_{\tau,m}$, then there is a polynomial p such that $s_{\mathscr{A},\tau,m} = \mu * \Phi_\tau + p$.

(c) The nonsingularity of the linear system (3.4) comes from Theorem 3.7 in [16] and the conditionally strictly positive definiteness of Φ_τ. $\square$

4. Error estimates and limit cases

In this section, we first give an estimation of the local error of interpolation. Let Ω be an open subset of $\mathbb{R}^d$. We say that Ω satisfies the cone property if there exist $r>0$ and $\theta>0$ such that for any $t\in\Omega$ there exists a unit vector $\xi(t)\in\mathbb{R}^d$ such that the cone

$$c(t,\xi(t),\theta,r) := \{t+\lambda\mu,\ \ \mu\in\mathbb{R}^d, ||\mu|| = 1, \mu\bullet\xi(t)\geqslant\cos\theta, 0\leqslant\lambda\leqslant r\}$$

is entirely contained within Ω.

Theorem 6. *Let Ω be an open, bounded connected subset of $\mathbb{R}^d$ having the cone property. For any $h>0$, let $\mathscr{A}_h$ be a finite subset of distinct points in Ω which contains a $\Pi_{m-1}(\mathbb{R}^d)$-unisolvent subset with $\sup_{t\in\bar{\Omega}}\inf_{a\in\mathscr{A}_h}||t-a|| = h$. For each $f\in\mathscr{H}_{\tau,m}$, let $s_{\mathscr{A}_h,\tau,m}$ be the unique element of $\mathscr{H}_{\tau,m}$ of minimal semi-norm $|\,.\,|_{\mathscr{H}_{\tau,m}}$ which coincides with f on $\mathscr{A}_h$. There exists a constant $c>0$, independent of h and τ, such that*

$$|f(x) - s_{\mathscr{A},\tau,m}(x)|\leqslant\frac{c}{\sqrt{\tau}}h|f|_{\mathscr{H}_{\tau,m}},\quad \text{for all } x\in\Omega.$$

Proof. The set Ω is an open, bounded connected subset of $\mathbb{R}^d$ having the cone property. Then, a direct application of Theorem 3.6 in [11] proves that there exists $h_0>0$ and constants $c_1,c_2>0$, both independent of h and ϕ_τ (this implies that c_1 and c_2 do not depend on τ), such that

$$|f(x) - s_{\mathscr{A}_h,\tau,m}(x)|^2\leqslant\frac{1}{2\tau^3}c_1^2\max_{0\leqslant t\leqslant c_2 h}|e^{-\tau t}+\tau t-1|\,|f|^2_{\mathscr{H}_{\tau,m}}$$

for all $x\in\Omega$ and $h<h_0$. Now,

$$\max_{0\leqslant t\leqslant c_2 h}|e^{-\tau t}+\tau t-1|\leqslant\tfrac{1}{2}(\tau c_2 h)^2.$$

Setting $c = \frac{c_1c_2}{2}$ gives the required result. □

Remark 3. The error estimate given by Theorem 6 suggests, as expected, that the pointwise error goes to 0 as $h\to 0$, namely the pointwise error goes to 0 when $\mathscr{A}_h$ becomes more and more dense in an open set Ω. Unfortunately, it does not give any hint of what happens when the parameter τ goes to 0 or to ∞, because of the factor $|f|_{\mathscr{H}_{\tau,m}}$ which depends on τ and whose behavior with τ is unknown.

It is why we now investigate the behavior of the radial basis function under tension when $\tau\to 0$ or $\tau\to\infty$. We first recall two examples of pseudo-polynomial $(m,s)-$splines [6]. Let m be an integer and s a real number such that $-m+\frac{d}{2}<s<\frac{d}{2}$, which is the required condition for the space of pseudo-polynomial splines to be a subspace of continuous functions on $\mathbb{R}^d$. The generating radial basis function of (m,s)–splines is

$$K_{m,s}(x) = \begin{cases} C_{m,s}||x||^{2m+2s-d}\ln(||x||) & \text{if } 2m+2s-d \in 2\mathbb{N}^*, \\ C_{m,s}||x||^{2m+2s-d} & \text{else,} \end{cases} \tag{4.1}$$

where $C_{m,s}$ is a known constant. The function $K_{m,s}$ generates a tempered distribution on $\mathbb{R}^d$ also denoted by $K_{m,s}$ satisfying on $\mathbb{R}^d$ the relation

$$\Delta^{m+s}K_{m,s} = \delta \quad (\text{Dirac's measure at the origin}), \tag{4.2}$$

where Δ^{m+s} is the Laplace operator of order $m+s$, with an appropriate extension of the usual iterated Laplacean operator to a real order, whenever s is a non integer number (see [3]).

Remark 4. Let us recall that the function given by (4.1) can be replaced by any other function which generates a tempered distribution on $\mathbb{R}^d$ satisfying, up to a multiplicative factor, relation (4.2).

Let $(q_1,\dots,q_{d+1})$ be the canonical basis of $\Pi_1(\mathbb{R}^d)$, with $q_j(x) = x^{(j)}$ for $j = 1,\dots,d$ and $q_{d+1}(x) = 1$ where $x = (x^{(1)},\dots,x^{(d)})^T \in \mathbb{R}^d$. We consider here two of the (m,s)–splines. The first one is for $m = 1$ and $s = (d-1)/2$. We obtain the $(1,(d-1)/2)$–spline which, in this case, is the pseudo-linear spline of the form $\sigma_\infty(x) = \sum_{i=1}^N a_i||x-x_i|| + a_{N+1}$, subject to $\sum_{i=1}^N a_i = 0$. For the second example, we choose $m = 2$ and $s = (d-1)/2$ again. We obtain the $(2,(d-1)/2)$–spline which, in this case, is the pseudo-cubic spline $\sigma_0(x) = \sum_{i=1}^N b_i||x-x_i||^3 + \sum_{i=1}^{d+1} c_i q_1(x)$, with the condition $\sum_{i=1}^N b_i p(x_i) = 0$, $\forall p \in \Pi_1(\mathbb{R}^d)$. Another interesting case of (m,s)–spline is obtained for the choice $m = 2$ and $s = (d-2)/2$, in this case, the $(2,(d-2)/2)$–spline is given by $\sigma_{\text{tps}}(x) = \sum_{i=1}^N d_i||x-x_i||^2\ln(||x-x_i||) + \sum_{i=1}^{d+1} e_i q_1(x)$, with the condition $\sum_{i=1}^N d_i p(x_i) = 0$, $\forall p \in \Pi_1(\mathbb{R}^d)$. In particular, we obtain in $\mathbb{R}^2$ the well-known thin plate spline (TPS), the term of TPS being justified by the fact that, in two dimension, the $(2,0)$–spline models some physical properties of thin plate. In the literature, many authors use also the term of TPS, for any $d \geqslant 2$.

Proposition 1. *Let $\mathscr{A} = \{x_1,\dots,x_N\}$ be a finite set of distinct points in $\mathbb{R}^d$ which contains a $\Pi_1(\mathbb{R}^d)$-unisolvent subset, let $f = (f_1,\dots,f_N)^T \in \mathbb{R}^N$ and let σ_0 and σ_∞ be respectively the pseudo-cubic and the pseudo-linear splines satisfying the interpolating conditions $\sigma_0(x_i) = \sigma_\infty(x_i) = s_{\mathscr{A},\tau,1}(x_i) = s_{\mathscr{A},\tau,2}(x_i) = f_i$ for $i = 1,\dots,N$. Then, for all $x \in \mathbb{R}^d$, $s_{\mathscr{A},\tau,2}(x)$ converges to $\sigma_0(x)$ as $\tau \to 0$ and $s_{\mathscr{A},\tau,1}(x)$ converges to $\sigma_\infty(x)$ as $\tau \to \infty$.*

Proof. Let $\Psi_\tau(x) = -\frac{1}{2\tau^3}(1 + \frac{1}{2}||\tau x||^2 - \frac{1}{6}||\tau x||^3)$. For a fixed $x \in \mathbb{R}^d$, by expanding the function Φ_τ, given by (3.2), we obtain

$$\Phi_\tau(x) = \Psi_\tau(x) + \frac{1}{2\tau^3}\sum_{k=4}^{\infty}\frac{(-\tau||x||)^k}{k!}.$$

The function $s_{\mathscr{A},\tau,2}$ is given by $s_{\mathscr{A},\tau,2}(x) = \sum_{i=1}^{N} \lambda_{i,\tau}\Phi_\tau(x-x_i) + \sum_{i=1}^{d+1} \alpha_{i,\tau}q_i(x)$, it follows that

$$s_{\mathscr{A},\tau,2}(x) = \sum_{i=1}^{N} \lambda_{i,\tau}\Psi_\tau(x-x_i) + \sum_{i=1}^{d+1} \alpha_{i,\tau}q_i(x) + \frac{1}{2\tau^3}\sum_{i=1}^{N}\lambda_{i,\tau}\left(\sum_{k=4}^{\infty}\frac{(-\tau||x-x_i||)^k}{k!}\right). \tag{4.3}$$

Let

$$E_\tau = \left(\frac{1}{2\tau^3}\sum_{k=4}^{\infty}\frac{(-\tau||x_i-x_j||)^k}{k!}\right)_{1\leqslant i,j\leqslant N}, \quad B_\tau = (\Psi_\tau(x_i-x_j))_{1\leqslant i,j\leqslant N}$$

and $A = (||x_i-x_j||^3)_{1\leqslant i,j\leqslant N}$ be the $N\times N$ matrices. The matrix A_τ given in the linear system (3.4), may be written as $A_\tau = B_\tau + E_\tau$. We have $||E_\tau||_\infty \to 0$ as $\tau\to 0$ and $\lambda_{i,\tau} \underset{\tau\to 0}{\sim} \gamma_{i,\tau}$, for $i=1,\dots,N$ and $\alpha_{j,\tau} \underset{\tau\to 0}{\sim} \beta_{j,\tau}$ for $j=1,\dots,d+1$, where the vectors $\gamma_\tau = (\gamma_{1,\tau},\dots,\gamma_{N,\tau})^T$ and $\beta_\tau = (\beta_{1,\tau},\dots,\beta_{d+1,\tau})^T$ are obtained by solving the following nonsingular linear system:

$$\begin{cases} B_\tau\gamma_\tau + M\beta_\tau &= f \\ M^T\gamma_\tau &= 0 \end{cases} \quad \text{as} \quad \tau\to 0. \tag{4.4}$$

The nonsingularity of the linear system (4.4) is guaranteed by the fact that the function Ψ_τ also generates a tempered distribution on $\mathbb{R}^d$ which is, up to a multiplicative factor, a fundamental solution of the operator Δ^{m+s} for $m=2$ and $s=(d-1)/2$. In fact, we have $\Delta^{\lfloor\frac{d-1}{2}\rfloor+2}[1+\frac{1}{2}||\tau x||^2] = \Delta^{\lfloor\frac{d-1}{2}\rfloor}[\Delta^2(1+\frac{\tau^2}{2}||x||^2)] = 0$, thus $\Delta^{\frac{d+3}{2}}\Psi_\tau = \frac{1}{12}\Delta^{\frac{d+3}{2}}[||x||^3] = \frac{1}{12C_{2,\frac{d-1}{2}}}\delta$, where $C_{2,\frac{d-1}{2}}$ is given in (4.1).

Let $B_\tau\gamma_\tau = ((B_\tau\gamma_\tau)_1,\dots,(B_\tau\gamma_\tau)_N)^T$; we have

$$(B_\tau\gamma_\tau)_j = \sum_{i=1}^{N}\gamma_{i,\tau}\Psi_\tau(x_j-x_i) = -\frac{1}{2\tau^3}\sum_{i=1}^{N}\gamma_{i,\tau}\left[1+\frac{\tau^2}{2}||x_j-x_i||^2 - \frac{\tau^3}{6}||x_j-x_i||^3\right].$$

Since $||x_j-x_i||^2 = ||x_j||^2 - 2x_j^Tx_i + ||x_i||^2$ and together with the orthogonality conditions $\sum_{i=1}^{N}\gamma_{i,\tau} = 0$ and $\sum_{i=1}^{N}\gamma_{i,\tau}x_i = 0$, we obtain that

$$(B_\tau\gamma_\tau)_j = \frac{1}{12}\sum_{i=1}^{N}\gamma_{i,\tau}||x_j-x_i||^3 - \theta_\tau, \quad \text{where } \theta_\tau = \frac{1}{4\tau}\sum_{i=1}^{N}\gamma_{i,\tau}||x_i||^2.$$

Let $e_{d+1} = (0,\dots,0,1)^T$, $b_\tau = \frac{1}{12}\gamma_\tau$ and $c_\tau = \beta_\tau - \theta_\tau e_{d+1}$, it follows that the linear system (4.4) becomes

$$\begin{cases} Ab_\tau + Mc_\tau &= f, \\ M^Tb_\tau &= 0. \end{cases} \tag{4.5}$$

The coefficients b_τ and c_τ obtained by solving the nonsingular linear system (4.5) are exactly the coefficients $b = (b_1, \dots, b_N)^T$ and $c = (c_1, \dots, c_{d+1})^T$ of the pseudo-cubic spline σ_0 satisfying the interpolating conditions $\sigma_0(x_i) = f_i$ for $i = 1, \dots, N$. It follows that $b_\tau \underset{\tau\to 0}{\sim} b$, $c_\tau \underset{\tau\to 0}{\sim} c$ and $\theta_\tau \underset{\tau\to 0}{\sim} \theta := \frac{3}{\tau}\sum_{i=1}^N b_i||x_i||^2$, which gives

$$\lambda_\tau \underset{\tau\to 0}{\sim} 12b \quad \text{and} \quad \alpha_\tau \underset{\tau\to 0}{\sim} c + \theta e_{d+1}. \tag{4.6}$$

From (4.3), again by using the relation $||x - x_i||^2 = ||x||^2 - 2x^T x_i + ||x_i||^2$ together with the orthogonality conditions $\sum_{i=1}^N \lambda_{i,\tau} = 0$ and $\sum_{i=1}^N \lambda_{i,\tau} x_i = 0$, we obtain

$$\begin{aligned} s_{\mathscr{A},\tau,2}(x) = & \frac{1}{12}\sum_{i=1}^N \lambda_{i,\tau}||x - x_i||^3 + \sum_{i=1}^{d+1} \alpha_{i,\tau} q_i(x) \\ & - \frac{1}{4\tau}\sum_{i=1}^N \lambda_{i,\tau}||x_i||^2 + \frac{1}{2\tau^3}\sum_{i=1}^N \lambda_{i,\tau}\left(\sum_{k=4}^{\infty} \frac{(-\tau||x - x_i||)^k}{k!}\right). \end{aligned} \tag{4.7}$$

Therefore, using (4.6), we obtain

$$s_{\mathscr{A},\tau,2}(x) \underset{\tau\to 0}{\sim} \sum_{i=1}^N b_i||x - x_i||^3 + \sum_{i=1}^{d+1} c_i q_i(x) + \frac{6}{\tau^3}\sum_{i=1}^N b_i\left(\sum_{k=4}^{\infty} \frac{(-\tau||x - x_i||)^k}{k!}\right),$$

namely,

$$s_{\mathscr{A},\tau,2}(x) \underset{\tau\to 0}{\sim} \sigma_0(x) + \frac{6}{\tau^3}\sum_{i=1}^N b_i\left(\sum_{k=4}^{\infty} \frac{(-\tau||x - x_i||)^k}{k!}\right).$$

On the other hand, when $\tau \to \infty$, we get $\Phi_\tau(x) \underset{\tau\to\infty}{\sim} \Phi_\infty(x) := -||\tau x||$ and we have $\Phi_\infty = cK_{1,\frac{d-1}{2}}$. Then by a similar argument, we obtain that $s_{\mathscr{A},\tau,1}(x) \underset{\tau\to\infty}{\sim} \sigma_\infty(x)$. $\square$

5. Numerical examples

In order to illustrate the behavior of the interpolant with our radial basis function under tension (RBFT), we include some examples. We fix $d = 2$ and we choose $m = 1$ and 2, respectively. The first example is the interpolation with RBFT at Franke's data which are scattered data points $x_i = (s_i, t_i)$ in the unit square $[0, 1] \times [0, 1]$ and f_i for $i = 1, \dots, N = 33$ are given values in $\mathbb{R}$. The Franke's data are taken from Franke [7] and Franke and Nielson [15, Table 1], they were used by the authors for testing their method of construction of interpolant surfaces with the concept of tension.

Figs. 1 and 2, show the behavior of the RBFT interpolating the Franke's data for small ($\tau = 0.1$), middle ($\tau = 50$) and large ($\tau = 1000$) values of the parameter of tension with $m = 1$ and 2, respectively. In order to have a comparison between the RBFT and the pseudo-polynomial splines, we gave, in Fig. 3, the plots of the thin plate spline, pseudo-cubic spline and pseudo-linear spline interpolating the same

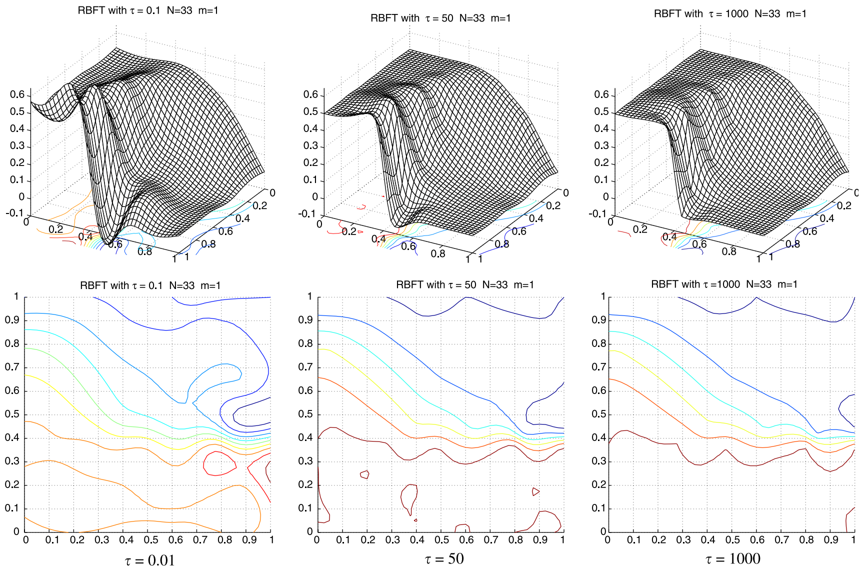

Fig. 1. RBFT interpolating the Franke's data with $N = 33$ and $m = 1$.

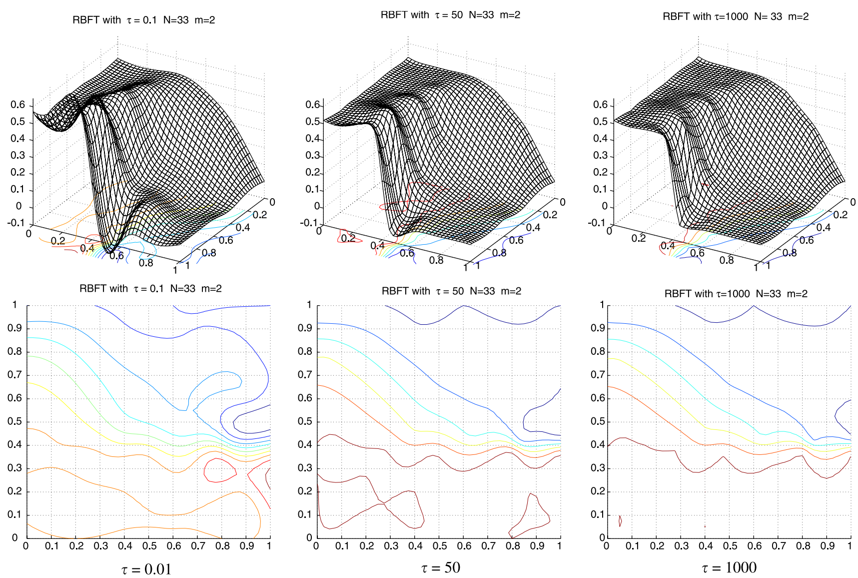

Fig. 2. RBFT interpolating the Franke's data with $N = 33$ and $m = 2$.

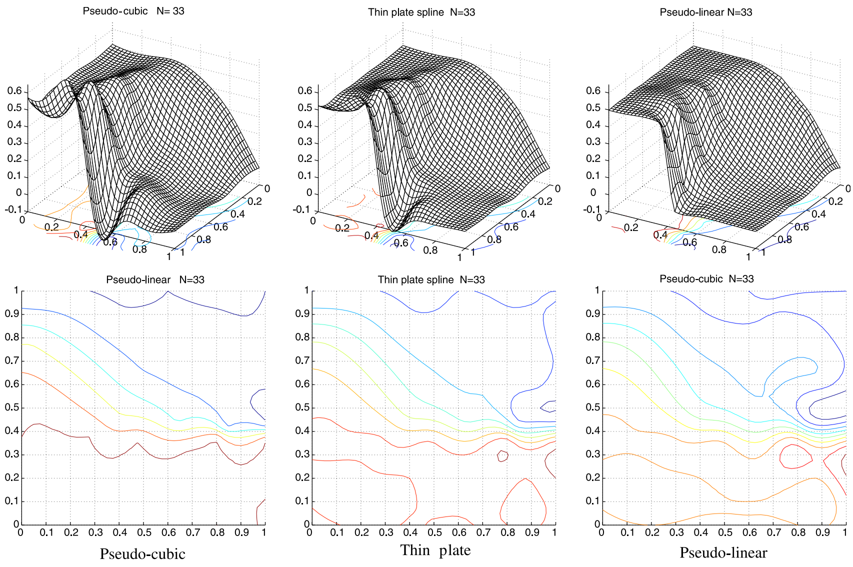

Fig. 3. Pseudo-polynomial splines interpolating the Franke's data with $N = 33$.

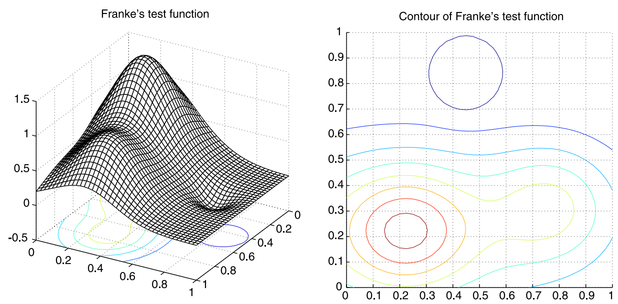

Fig. 4. Franke's test function and its contour.

data together with RBFT. This example illustrates the practical effects of the parameter τ of tension on the behaviour of the surface.

The second example is the interpolation of the usual well-known test function [15], given below. Fig. 4, shows the plot of the test function

$$f(x,y) = \tfrac{3}{4} e^{-\frac{(9x-2)^2+(9y-2)^2}{4}} + \tfrac{3}{4} e^{-\frac{(9x-1)^2}{49} - \frac{(9y-1)^2}{10}} - \tfrac{2}{10} e^{-(9x-4)^2-(9y-7)^2} + \tfrac{1}{2} e^{-\frac{(9x-7)^2+(9y-3)^2}{4}}.$$

We randomly generated scattered data points $x_i = (s_i, t_i)$ for $i = 1, \ldots, N$ in the unit square domain $[0,1] \times [0,1]$ and test the interpolation of the test function for different values of τ and different number N of scattered data points. For the sake of

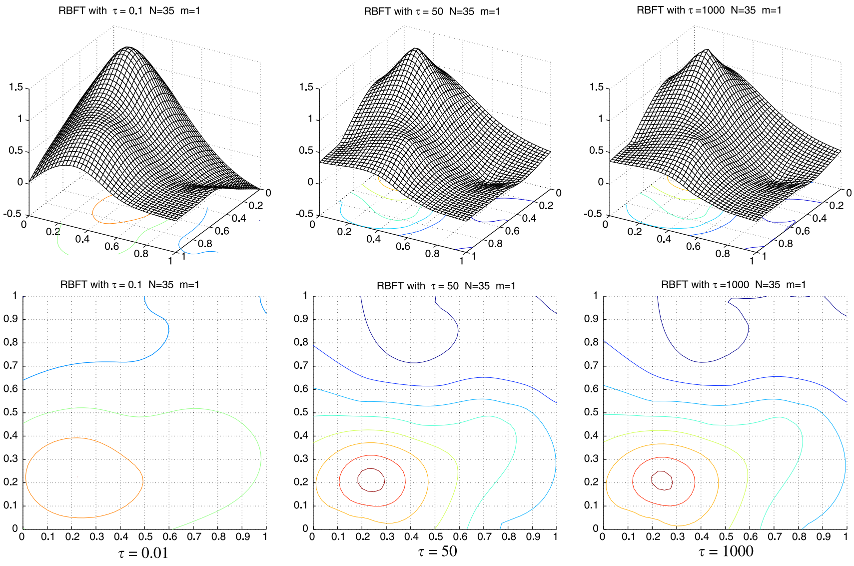

Fig. 5. RBFT interpolating f with $N = 35$ and $m = 1$.

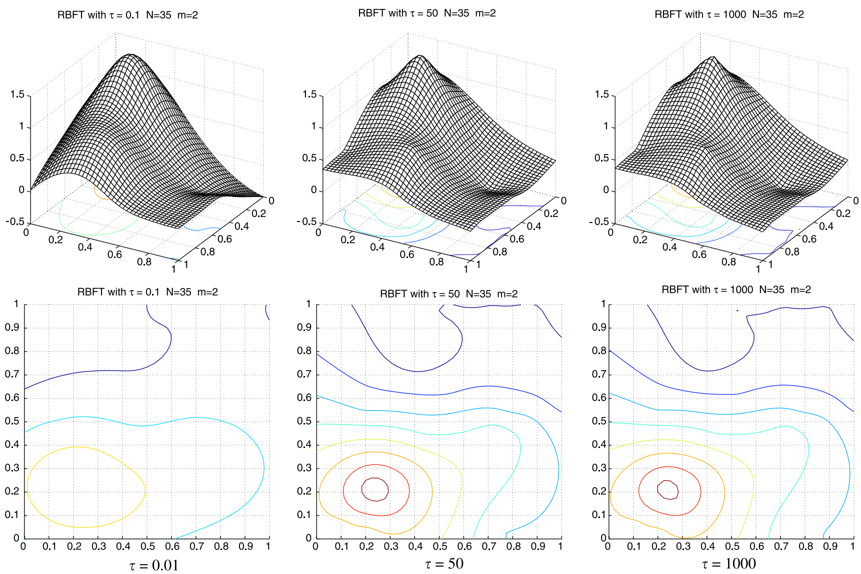

Fig. 6. RBFT interpolating f with $N = 35$ and $m = 2$.

brevity, we gave here only two values of N, one of a small value ($N = 35$) and the other one of a large value ($N = 2000$). We chose the values of the parameter of tension, to be small ($\tau = 0.01$), middle ($\tau = 50$) and large ($\tau = 1000$), respectively.

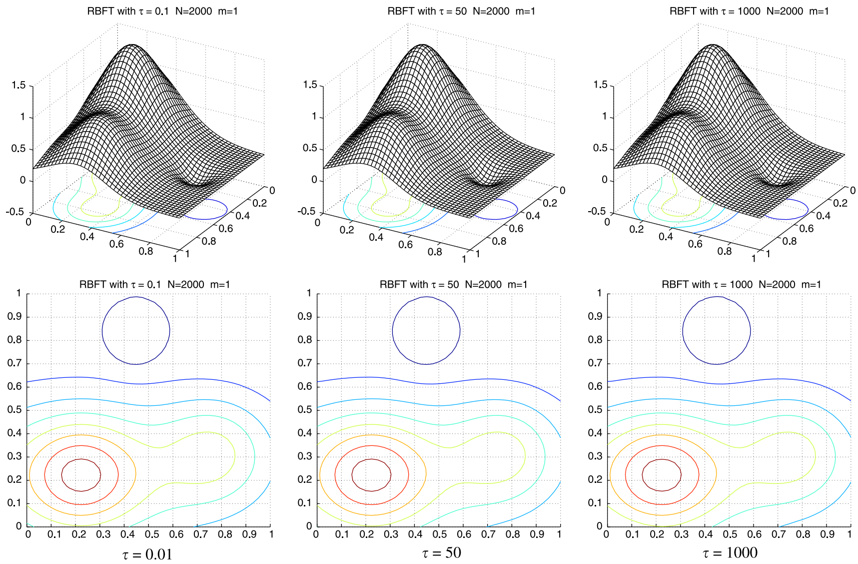

Fig. 7. RBFT interpolating f with $N = 2000$ and $m = 1$.

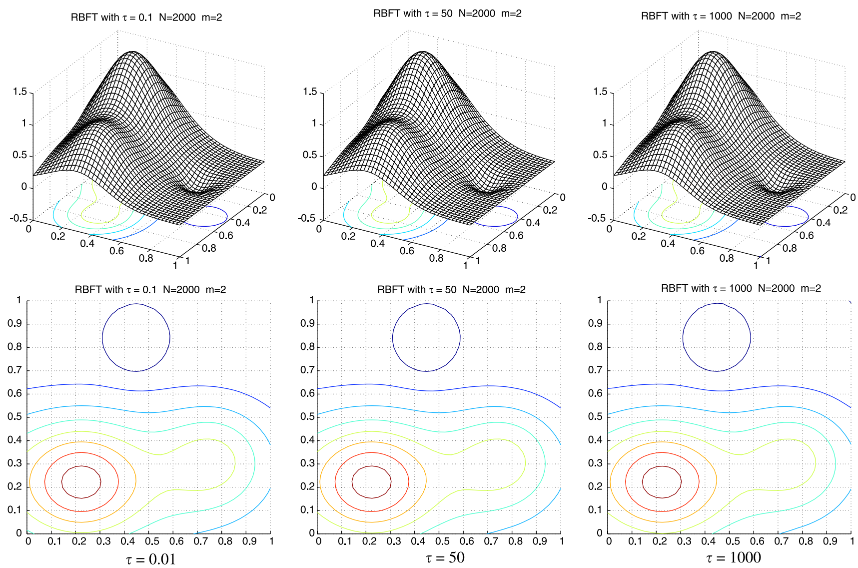

Fig. 8. RBFT interpolating f with $N = 2000$ and $m = 2$.

Figs. 5 and 7, on one hand, and Figs. 6 and 8 on the other hand, show the behavior of the RBFT interpolation for $m = 1$ and 2, respectively.

We observe that in both cases ($m = 1$ and 2), the RBFT behave like the pseudo-cubic spline when τ is small, and like the pseudo-linear spline for large value of τ. We

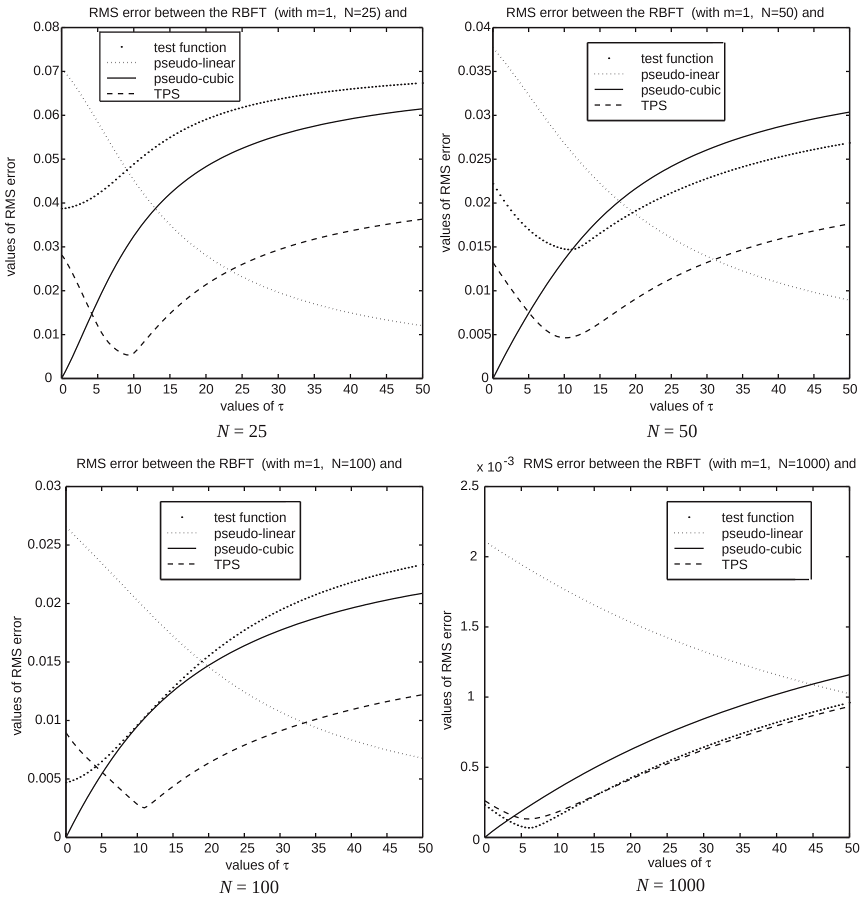

Fig. 9. RMS errors between the RBFT (with $m = 1$) and the test function, together with the pseudo-polynomial splines.

have noticed that the parameter τ does not have any visual effect when N becomes large, which is a direct consequence of the fact that the RBFT, like the pseudo-polynomial splines, converges to the underlying function f when the set of the interpolating points becomes more and more dense in an open set Ω, which is the main assertion of Theorem 6. As we were able to observe, there is only a slight difference between the two cases $m = 1$ or $m = 2$. Choosing $m = 1$ or $m = 2$, does not either seriously affect the behavior of the RBFT nor seriously modify the accuracy of the interpolation.

In order to provide some additional light on the effects of the parameter τ, we examined the root-mean-square (RMS) errors between the RBFT and the function test together with the pseudo-polynomial splines, for various value of τ. We choose an uniform grid of $n = 100 \times 100$ points (u_i, v_i) on $[0, 1] \times [0, 1]$, the $RMS(g, h)$ error

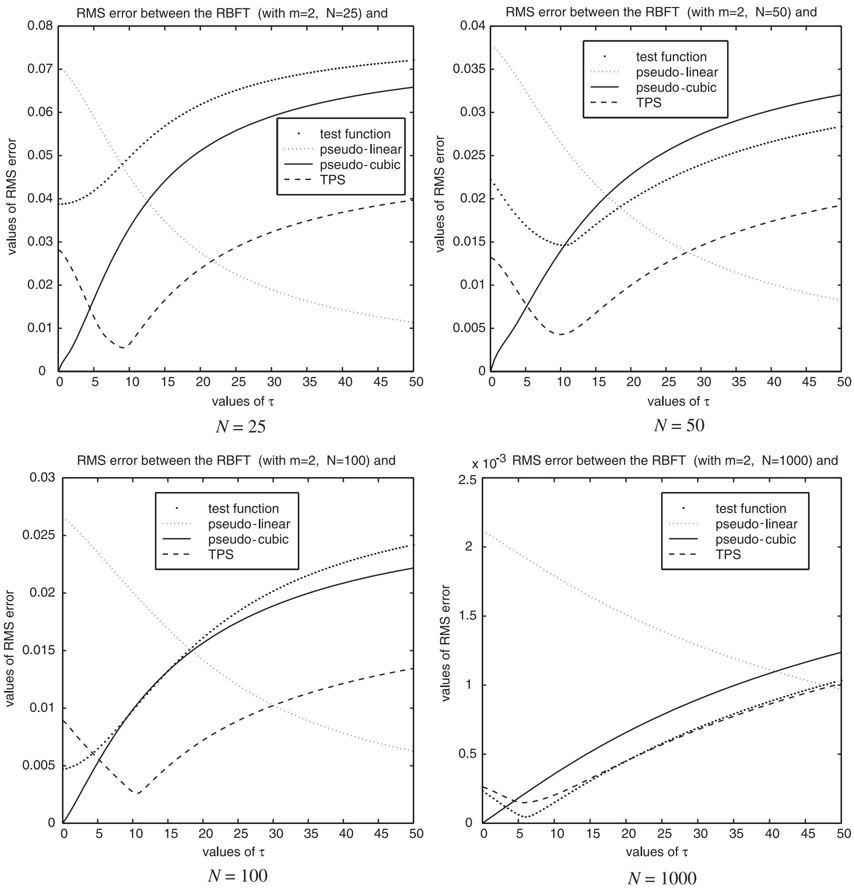

Fig. 10. RMS errors between the RBFT (with $m = 2$) and the test function, together with the pseudo-polynomial splines.

between two functions g and h is computed at the grid points by

$$(RMS)(g,h) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} |g(u_i,v_i) - h(u_i,v_i)|^2}.$$

In Figs. 9 and 10, we have plotted, the root-mean-square errors $\tau \to (RMS)(f, s_{\mathscr{A},\tau,m})$, $\tau \to (RMS)(\sigma_0, s_{\mathscr{A},\tau,m})$, $\tau \to (RMS)(\sigma_\infty, s_{\mathscr{A},\tau,m})$ and $\tau \to (RMS)(\sigma_{\rm tps}, s_{\mathscr{A},\tau,m})$ as functions involving the values of τ, where $s_{\mathscr{A},\tau,m}$, σ_0, σ_∞ and $\sigma_{\rm tps}$ are the RBFT, the pseudo-cubic, the pseudo-linear and the TPS interpolating the test function f on a same set $\mathscr{A}$ of N scattered data points. The value of the parameter τ is incremented from 0.1 to 50 and the curves of Figs. 9 and 10 are computed point by point. We observe that, $(RMS)(\sigma_0, s_{\mathscr{A},\tau,m})$ and $(RMS)(\sigma_\infty, s_{\mathscr{A},\tau,m})$ become more and more small as τ becomes more and more small and large, respectively.

Table 1
The RMS error for RBFT with optimal τ

$m=1$			$m=2$		
N	τ_{optimal}	RMS error	N	τ_{optimal}	RMS error
25	0.01	3.8711e-2	25	0.01	3.8710e-2
50	11.01	1.4703e-2	50	10.51	1.4606e-2
100	0.01	4.6971e-3	100	0.01	4.6963e-3
1000	6.011	6.8685e-5	1000	6.01	4.6449e-5

Table 2
The RMS error for Pseudo-polynomial splines

Pseudo-cubic		Thin plate		Pseudo-linear	
N	RMS error	N	RMS error	N	RMS error
25	3.8709e-2	25	4.9840e-2	25	7.2336e-2
50	2.2211e-2	50	1.7708e-2	50	3.4302e-2
100	4.6955e-3	100	1.2055e-2	1000	2.9474e-2
1000	2.3199e-4	100	1.8331e-4	1000	1.9306e-3

A critical problem using the RBFT method is obviously in the evaluation of an optimal tension parameter τ or at least to find a way which allows the user to choose a suitable value of the tension parameter in order to obtain an expected behavior for the resulting surface. This certainly depends on the geometry of the scattered data interpolating points. It seems also that some cross validation technique might help in this case and this will be investigated elsewhere. The Table 1 gives an experimental estimation of the optimal tension parameter τ_{opt} together with the corresponding $(RMS)(f, s_{\mathscr{A},\tau_{\text{opt}},m})$ error. The Table 2, gives the $(RMS)(f,\sigma_0)$, $(RMS)(f,\sigma_\infty)$ and $(RMS)(f,\sigma_{\text{tps}})$ errors for the pseudo-polynomial splines. We observed, that the RBFT with the empirical optimal value of the parameter τ has a better RMS error than the pseudo-polynomial splines. It is also interesting to note that there is a value of τ for the $(RMS)(\sigma_{\text{tps}}, s_{\mathscr{A},\tau,m})$ error to be minimal, namely there is a value of τ such that the RBFT gives a "best approximation" of the TPS.

Acknowledgments

The authors express their thanks to the anonymous referees for their interesting suggestions, useful comments and careful reading of this paper, which helped to improve the presentation of this paper.

References

[1] A. Bouhamidi, Hilbertian Approach for univariate spline with tension, J. Approx. Theory Appl. 17 (4) (2001) 36–57.
[2] A. Bouhamidi, A. Le Méhauté, Spline curves and surfaces with tension, in: P.J. Laurent, A. Le Méhauté, L.L. Schumaker (Eds.), Wavelets, Images, and Surface Fitting, A.K. Peters, Wellesley, 1994, pp. 51–58.
[3] A. Bouhamidi, A. Le Méhauté, Multivariate interpolating (m, ℓ, s)–splines, Adv. Comput. Math. 11 (1999) 287–314.
[4] M.D. Buhmann, Radial basis functions, Acta Numer. (2000) 1–38.
[5] J. Duchon, Interpolation de fonctions de deux variables suivant le principe de la flexion des plaques minces, RAIRO Anal. Numér. 10 (1975) 5–12.
[6] J. Duchon, Splines minimizing rotation-invariant seminorms in Sobolev spaces, in: W. Schempp, K. Zeller (Eds.), Constructive Theory of Functions of Several Variables, Lecture Notes in Mathematics, Vol. 571, Springer, Berlin, 1977, pp. 85–100.
[7] R. Franke, Thin plate splines with tension, Comput. Aided Geom. Design 2 (1985) 87–95.
[8] K. Guo, S. Hu, X. Sun, Conditionally positive definite functions and Laplace–Stieltjes integrals, J. Approx. Theory 74 (1993) 249–265.
[9] R.L. Harder, R.N. Desmarais, Interpolation using surface splines, J. Aircraft 9 (1972) 189–191.
[10] R.L. Hardy, Multiquadric equations of topography and other irregular surfaces, J. Geophys. Res. 76 (1971) 1905–1915.
[11] W. Light, H. Wayne, On power functions error estimates for radial basis function interpolation, J. Approx. Theory. 92 (1998) 245–266.
[12] W.R. Madych, S.A. Nelson, Multivariate interpolation and conditionally positive definite functions, J. Approx. Theory Appl. 4 (1988) 77–89.
[13] W.R. Madych, S.A. Nelson, Multivariate interpolation and conditionally positive definite functions II, Math. Comput. 54 (1990) 211–230.
[14] C.A. Micchelli, Interpolation of scattered data: distance matrices and conditionally positive definite functions, Constr. Approx. 2 (1986) 11–22.
[15] G.M. Nielson, R. Franke, A method for construction of surfaces under tension, Rocky Mountain J. Math. 14 (1) (1984).
[16] M.J.D. Powell, The theory of radial basis function approximation in 1990, in: W.A. Light (Ed.), Advances in Numerical Analysis: Wavelets, Subdivision, and Radial Functions, Oxford University Press, Oxford, 1992, pp. 105–210.
[17] R. Schaback, Multivariate interpolation and approximation by translates of a basis function, in: C.K. Chui, L.L. Schmaker (Eds.), Approximation Theory VIII, Vol. 1, World Scientific, Singapore, 1995, pp. 491–514.
[18] L. Schwartz, Sous-espace hilbertiens d'espaces vectoriels topologiques et noyaux associés (noyaux reproduisants), J. Anal. Math. 13 (1964) 115–256.
[19] D.G. Schweikert, An interpolation curve using splines in tension, J. Math. Phys. 45 (1966) 312–317.